Photogrammetry & Robotics Lab

On the role and evaluation of rigorous and approximate estimation methods

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Motivation

How much would you pay more/less if

- the accuracy (stdv) of an *instrument* halves/doubles?
- an *algorithm* leads to
  half/double the accuracy of its result?
- if an *algorithm* provides the uncertainty of the result?
- If an *algorithm* is faster/slower by a factor 2?
- If CPU-time of an algorithm is predictable?

Notions

- Estimation methods: observations → parameters
  (least squares, maximum likelihood, …)
- Rigorous: following some optimization principle
- Approximate: simplified, suboptimal solutions
  → e.g. hierarchy of approximations $\Sigma \longrightarrow \mathrm{Diag}(\sigma_n^2) \longrightarrow \sigma^2 I_N$
- Role: approximations often are fast and cheap
- Evaluation
  - Comparison: degree of loss in accuracy, gain in speed
  - Uncertainty of suboptimal methods

1. Example: Motion from point pairs

Motion from point pair

Given: set of point pairs $\{\boldsymbol{x}_i, \boldsymbol{x}'_i\}, i = 1, ..., I$

Assumption: related by rigid motion $\boldsymbol{x}'_i = R\boldsymbol{x}_i + \boldsymbol{t}$

Task: Estimate motion $(\widehat{R}, \widehat{\boldsymbol{t}})$

Classical solution (Arun et al. 1987)

- Centre data
- SVD → rotation
- translation

Arun's solution

1. Centre data in both systems

$$\bar{\boldsymbol{x}}_i = \boldsymbol{x}_i - \boldsymbol{x}_C, \quad \text{with} \quad \boldsymbol{x}_C = \sum_i \boldsymbol{x}_i / I$$

$$\bar{\boldsymbol{x}'}_i = \boldsymbol{x}'_i - \boldsymbol{x}'_C, \quad \text{with} \quad \boldsymbol{x}'_C = \sum_i \boldsymbol{x}'_i / I$$

2. Determine rotation

$$UDV = \mathrm{SVD}(\textstyle\sum_i \bar{\boldsymbol{x}}_i \bar{\boldsymbol{x}'_i}) \qquad R = V\ \mathrm{Diag}(1, 1, \det(UV)\ U^{\mathsf{T}}$$

3. Estimated motion $(R, \boldsymbol{x}'_C - R\boldsymbol{x}_C)$

Is Arun's method optimal/rigorous?

Pro:

- It minimizes the sum of the squared distances (LS)
- If all coordinates have the same accuracy (ML)

Contra:

- Real data have no homogeneous accuracy
- Not statistically rigorous: uncertainty of data unused

→ ?

Outline

1. An example: motion from point pairs
2. Statistically optimal estimation
   a. ML estimation
   b. Visualization, effect of CovM
3. Uncertainty of approximate estimates
4. Characterizing accuracy loss
5. Examples
   a. Motion from point and plane pairs
   b. Bundle adjustment
6. Closing

2. Statistically optimal estimation

Statistically optimal estimation (ML)

Maximum likelihood estimation

(my teacher said: method for self-confident people)

- Data are sample from specified distribution
- Distribution is parametrized
- Parameters are functionally related by constraints
- Estimated parameters give best explanation of observations

Formal setup of ML-estimation

- Observations: $\boldsymbol{l} = [l_n], n = 1, ..., N$
- Distribution: Gaussian $\underline{\boldsymbol{l}} \mid \boldsymbol{y}, \sigma_0^2 \Sigma_{ll}^a \sim \mathcal{N}(\boldsymbol{y}, \sigma_0^2 \Sigma_{ll}^a)$

  or

  $$p(\boldsymbol{l} \mid \boldsymbol{y}, \sigma_0^2 \Sigma_{ll}^a) = k \exp\left(-(\boldsymbol{l} - \boldsymbol{y})^{\mathsf{T}} (\sigma_0^2 \Sigma_{ll}^a)^{-1} (\boldsymbol{l} - \boldsymbol{y})/2\right)$$

  parameters:

  - $\boldsymbol{y}$ unknown mean observation (expected value)
  - Σ_{ll}^a : given approximate covariance matrix
  - σ_0^2 : unknown variance factor
- Constraints with additional unknown parameters $\boldsymbol{x}$

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}$$

Task

Optimization task: find values $\boldsymbol{x}$ and $\boldsymbol{y}$, which

$$\text{minimize} \quad (\boldsymbol{l}-\boldsymbol{y})^{\mathsf{T}} W_{ll}^{a}(\boldsymbol{l}-\boldsymbol{y}) \quad \text{with} \quad W_{ll}^{a} = (\Sigma_{ll}^{a})^{-1}$$

$$\text{subject to} \quad \boldsymbol{g}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{0}$$

Statistically optimal method

Input: observations $\boldsymbol{l}$

Method:

- parameters of distribution
- constraints
- Optimization criterion

Output: [$\widehat{\sigma}_0^2$, $(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}})$, $\Sigma_{\widehat{x}\widehat{x}}$]

- Estimate $\widehat{\sigma}_0^2$ for variance factor → model fit
- Estimates $(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}})$ for unknown parameters,

  →Corrections $\widehat{\boldsymbol{v}} = \widehat{\boldsymbol{y}} - \boldsymbol{l}$, residuals/errors $\widehat{\boldsymbol{e}} = \boldsymbol{l} - \widehat{\boldsymbol{y}} = -\widehat{\boldsymbol{v}}$
- predicted covariance matrix $\Sigma_{\widehat{x}\widehat{x}}$ of $\widehat{\boldsymbol{x}}$

Solution for linear constraints

For linear model

$$\boldsymbol{g}(\boldsymbol{x},\boldsymbol{y}) = A\boldsymbol{x} + B^{\mathsf{T}}\boldsymbol{y} + \boldsymbol{b} = \boldsymbol{0}$$

Use substitute corrections

$$\boldsymbol{n} = B^{\mathsf{T}}\boldsymbol{l} + \boldsymbol{b} \quad \text{and} \quad \mathbb{D}(\underline{\boldsymbol{n}}) = \Sigma_{nn} = B^{\mathsf{T}}\Sigma_{ll}B$$

Estimates

$$\widehat{\boldsymbol{x}} = \ -(A^{\mathsf{T}}\Sigma_{nn}^{-1}A)^{-1}\ A^{\mathsf{T}}\Sigma_{nn}^{-1}\ \boldsymbol{n}(\boldsymbol{l})$$

$$\widehat{\boldsymbol{y}} = \boldsymbol{l} - \Sigma_{ll}B\Sigma_{nn}^{-1}\ \boldsymbol{g}(\widehat{\boldsymbol{x}},\boldsymbol{l})$$

Covariance matrix

$$\Sigma_{\widehat{x}\widehat{x}} = (A^{\mathsf{T}}(B^{\mathsf{T}}\Sigma_{ll}B)^{-1}A)^{-1}$$

Visualization of ML estimation

Stochastical models:

$$\underline{\boldsymbol{l}} \sim \mathcal{N}(\boldsymbol{a}x, I_2)$$

$$\underline{\boldsymbol{l}} \sim \mathcal{N}(\boldsymbol{a}x, \Sigma)$$

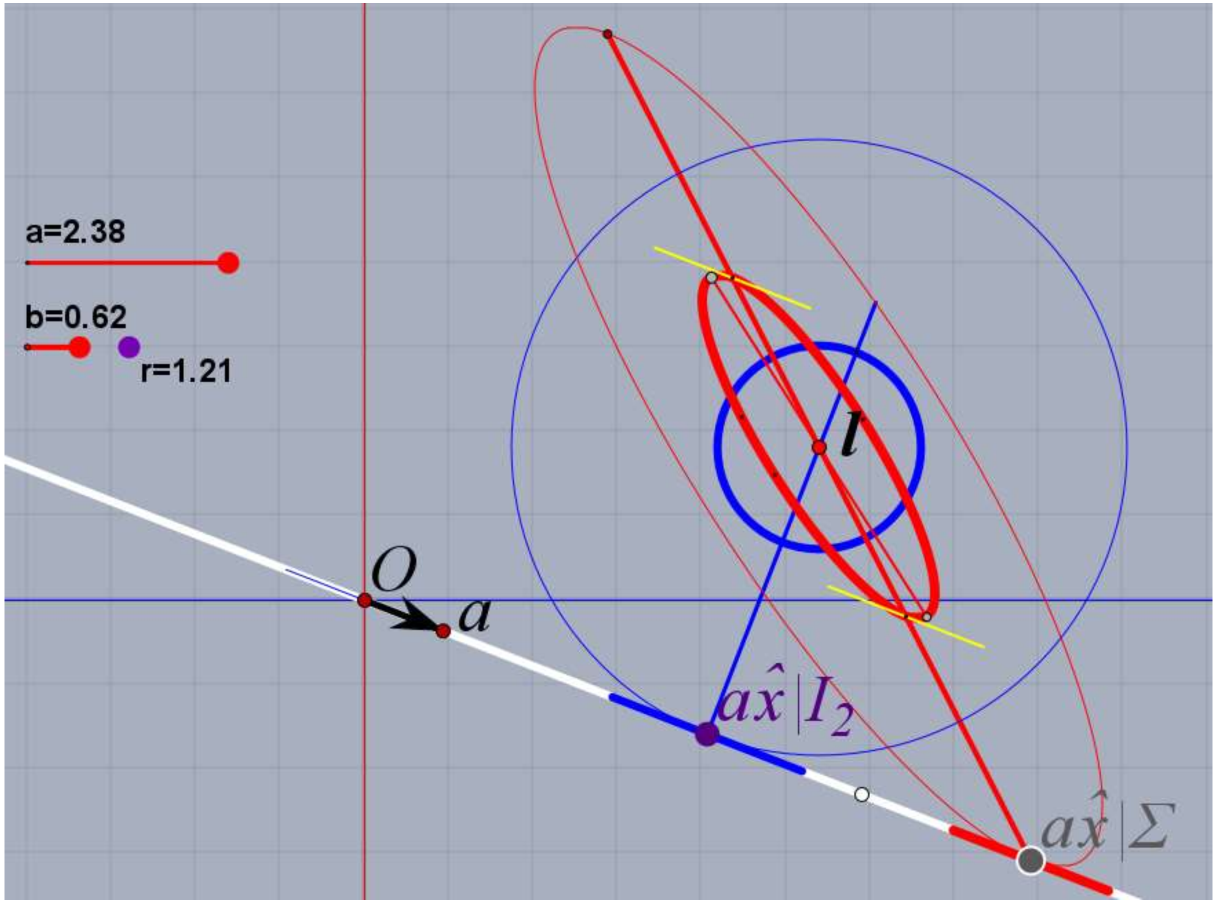
Constraint (white line)

$$\boldsymbol{a}x - \boldsymbol{y} = \boldsymbol{0}$$

Estimates

$$\widehat{\boldsymbol{a}x} \mid I_2$$

$$\widehat{\boldsymbol{a}x} \mid \Sigma$$



(which is rigorous/optimal?)

Results of ML estimation

Estimates

$\widehat{ax} \mid I_2$ $\widehat{ax} \mid \Sigma$

Covariance matrix

(here of $\widehat{ax}$)

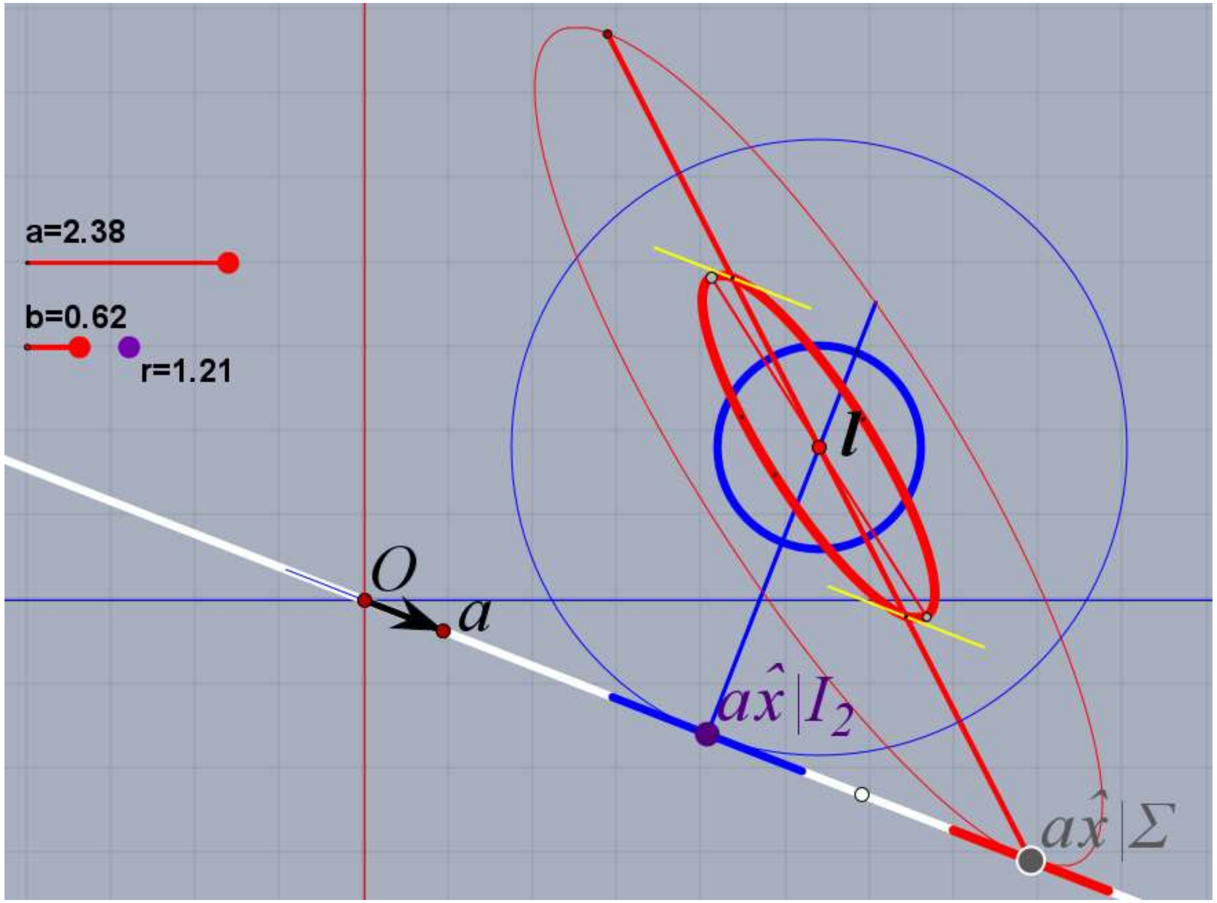
blue and red segments

(flat/singular ellipses)

(sqrt) variance factors

ratio of thin and thick

standard ellipses



Exploring extreme cases

- Can changing the covariance matrix have zero effect?
- Can the mean of 0 and 1 be -1?
- Can for given observations and constraints the unknown parameters be arbitrary?

Cdy-app

Result of exploration

- Changing the model may have no effect
- You can find a model for each result

→We need to carefully specify (common sense!)

Questions:

- How to compare quality of methods?
- Can we do the same with approximate methods?
  (eg. those using SVD or with complex algebraic derivation?

Type of approximations

Approximations in estimation methods

Motivation for using approximations:

- Computational efficiency
- Lack of knowledge about uncertainty, systematic errors, internals of a method, …

Examples for approximations

- Using a non-statistical approach

  e.g. minimizing residuals of constraints (algebraic minim.)
- Early stopping of iteration process

  e.g. a single iteration, assuming good approximate values

  → Fixed computing time
- Simplifying CovM of the observations

  e.g. neglecting correlations, assuming constant stdv.
- Simplifying Jacobians J

  e.g. evaluating at non-optimal point, fixing J after 1. iter.

Minimal and closed form solutions

Examples

- 5 point algorithm for essential matrix
- 5 point algorithm for cylinder
- 7 point algorithm for fundamental matrix
- Plane estimation from point clouds
- Motion estimation from point pairs
- Motion estimation from plane pairs
- Homography estimation from points

See http://aag.ciirc.cvut.cz/minimal/ (172 entries)

Questions

- Can we provide uncertainty for approximate solutions?

  → would allow to compare CovM of methods

Two paths

1. Perform simulations → potentially high effort
2. Derive algebraic expression → short cut

   (may be only for class of approximations)

3. Uncertainty of approximate estimates

a. from simulations

b. from algebra

a. Empirical covariance matrix

Method M (black box)

$$M: \quad \boldsymbol{l} \mapsto \boldsymbol{x}(\boldsymbol{l})$$

1. Specify experimental design:
   - ideal, consistent

     $$(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{l}}) \text{ consistent with } \boldsymbol{k}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{l}}) = \boldsymbol{0}$$

   - Assume uncertainty Σ_{ll} of observations
2. Sample J observations

   $$\boldsymbol{l}_j \sim \mathcal{N}(\tilde{\boldsymbol{l}}, \Sigma_{ll})$$

3. Derive empirical mean and CovM of $\widehat{\boldsymbol{x}}_j = \boldsymbol{x}(\boldsymbol{l}_j)$

… Choice of Design

- Free choice of parameters and CovM
- Use results of real case (user friendly)

  estimated parameters: $\tilde{\boldsymbol{x}} := \widehat{\boldsymbol{x}}$

  Fitted observations: $\tilde{\boldsymbol{l}} := \boldsymbol{l} + \widehat{\boldsymbol{v}}$

- Parametrize experimental design

  e.g. depending on

  # images (T), focal length (f), #points (I), overlap (o), …

$$\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{l}}(T, f, I, o) \quad \text{and} \quad \tilde{\boldsymbol{l}} = \tilde{\boldsymbol{l}}(T, f, I, o) \mapsto \Sigma_{\widehat{x}\widehat{x}}(T, f, I, o)$$

(algebraic derivation)

CovM for minimal solutions

2a. CovM for minimal solutions (1/2)

Determine U parameters $\boldsymbol{x}$ from U observations

Assumption for observed values

$$\boldsymbol{y} \sim \mathcal{M}(\mathbb{E}(\underline{\boldsymbol{l}}), \Sigma_{ll}) \quad \text{with} \quad \underline{\boldsymbol{l}} + \underline{\boldsymbol{v}} = \mathbb{E}(\underline{\boldsymbol{l}})$$

Constraints

- between parameters and observations (G)
- among parameters only (H)

$$\underset{U\times 1}{\boldsymbol{k}(\boldsymbol{x},\boldsymbol{y})} = \begin{bmatrix} \underset{G\times 1}{\boldsymbol{g}(\boldsymbol{x},\boldsymbol{y})} \\ \underset{H\times 1}{\boldsymbol{h}(\boldsymbol{x})} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

a1. CovM for minimal solutions (2/2)

Direct solution, may be complex, multiple parameters

$$\boldsymbol{x}_t = \boldsymbol{f}_t(\boldsymbol{l})\,, t = 1, ..., T$$

But: covariance matrix of each pair $(\boldsymbol{x}_t, \boldsymbol{l}), t = 1, ..., T$

$$\boxed{\Sigma_{x_t x_t} = A_t^{-1} B_t^{\mathsf{T}} \Sigma_{ll} B_t A_t^{-\mathsf{T}}}$$

with

$$\underset{U \times U}{A}(\boldsymbol{x}, \boldsymbol{l}) = \left.\frac{\partial \boldsymbol{k}}{\partial \boldsymbol{x}^{\mathsf{T}}}\right|_{x,l} \quad \text{and} \quad \underset{U \times N}{B^{\mathsf{T}}}(\boldsymbol{x}, \boldsymbol{l}) = \left.\frac{\partial \boldsymbol{k}}{\partial \boldsymbol{l}^{\mathsf{T}}}\right|_{x,l}$$

Example: Fundamental matrix F

Matrix F from 7 point pairs $[\mathbf{x}'; \mathbf{x}'']_i$, 42 observations

$$\underset{7\times 1}{\boldsymbol{g}} = [\mathbf{x}_i{'}^{\mathsf{T}} \mathsf{F} \mathbf{x}_i''] = \mathbf{0} \quad \text{and} \quad \underset{2\times 1}{\boldsymbol{h}} = \begin{bmatrix} ||\mathsf{F}|| - 1 \\ \det(\mathsf{F}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Minimal solution with SVD, leads to up to 3 pairs

$$(\mathsf{F}_t, [[\mathbf{x}', \mathbf{x}'']_i]), t = 1, 2, 3$$

Jacobians wrt 9 parameters $\mathbf{f} = \text{vec}\mathsf{F}$ and obserations

$$\underset{9\times 9}{A} = \begin{bmatrix} [\mathbf{x}_i{''}^{\mathsf{T}} \otimes \mathbf{x}_i^{\mathsf{T}}] \\ \text{vec}^{\mathsf{T}}(\mathsf{F}) \\ \text{vec}^{\mathsf{T}}(\mathsf{F}^{\mathrm{O}}) \end{bmatrix}, \qquad \underset{9\times 42}{B} = \begin{bmatrix} I_7 \otimes \text{vec}^{\mathsf{T}}(\mathsf{F}) \\ \mathbf{0}^{\mathsf{T}} \\ \mathbf{0}^{\mathsf{T}} \end{bmatrix}$$

(cofactor matrix F^{O})

Insight

- In spite of algebraically complex solution
  comparably simple variance propagation

- Correct rank 7 of estimated covariance matrix
  since rank of B is 7

- Assumptions for general cases
  - Only U constraints are used
  - Minimal solution $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{l})$ is differentiable (no decisions)

CovM for direct non-minimal solutions

a2. CovM for direct non-minimal solutions

Determine U parameters $\boldsymbol{x}$ from N observations $\boldsymbol{l}$

Class of solutions: G constraints linear in parameters

$$\boldsymbol{g}(\mathbf{x}, \boldsymbol{y}) = \underset{G \times U}{A(\boldsymbol{y})}\, \mathbf{x} = \mathbf{0} \quad \text{and} \quad |\mathbf{x}|^2 = 1 \quad \text{with} \quad \text{rk}(A) = U - 1$$

Example: homography matrix form > 8 point pairs

a2. CovM for direct non-minimal solutions

Residuals

$$\boldsymbol{g} = A(\boldsymbol{l})\boldsymbol{x} \neq \boldsymbol{0}$$

Minimize sum of squared residuals

$$\widehat{\boldsymbol{x}} = \min_{x,|x|=1} \boldsymbol{g}(\boldsymbol{x})^{\mathsf{T}} \boldsymbol{g}(\boldsymbol{x})$$

LS solution with one constraint

Covariance matrix of estimate $\widehat{\boldsymbol{x}}$?

Interprete solution as (quasi) Gauss-Markov model

Starting from mean/true value $\tilde{\boldsymbol{x}}$, thus

$$\mathbf{x} = \widetilde{\mathbf{x}} + \Delta\mathbf{x} \quad \text{and} \quad \boldsymbol{l} = \boldsymbol{y} + \boldsymbol{v}$$

Linearize constraints $\quad \boldsymbol{g} = \boldsymbol{g}(\mathbf{x}, \boldsymbol{l}) = A(\boldsymbol{l})\,\mathbf{x}$

$$A(\boldsymbol{l})\,\mathbf{x} = A(\boldsymbol{y})\,\tilde{\mathbf{x}} + A(\boldsymbol{y})\,\Delta\mathbf{x} - B^{\mathsf{T}}(\tilde{\mathbf{x}}, \boldsymbol{y})\,\underline{\boldsymbol{v}}$$

With

$$A(\boldsymbol{y})\mathbf{x} = \mathbf{0} \quad \text{and} \quad \boldsymbol{v}_g = B^{\mathsf{T}}(\tilde{\mathbf{x}}, \boldsymbol{y})\boldsymbol{v}$$

We obtain linearized model (GM-model)

$$\boldsymbol{g} + \boldsymbol{v}_g = A\Delta\,\mathbf{x} \quad \text{with} \quad \mathbf{x}^{\mathsf{T}}\Delta\mathbf{x} = 0$$

Solution of GM model

…after some steps (see text)

$$\widehat{\Delta \mathbf{x}} = A_1^+ \boldsymbol{g}$$

With rank constrained matrix

$$A_1 = UD_1V^\mathsf{T}, D_1 = \mathrm{Diag}([d_1, ..., d_{U-1}, 0]) \quad \text{with} \quad A(\boldsymbol{l}) = UDV^\mathsf{T}$$

Covariance matrix

$$\Sigma_{\widehat{xx}} = A_1^+ \Sigma_{gg} A_1^{+\mathsf{T}} \quad \text{with} \quad \Sigma_{gg} = B^\mathsf{T} \Sigma_{ll} B$$

Thus finally

$$\boxed{\Sigma_{\widehat{xx}} = A_1^+ B^\mathsf{T} \Sigma_{ll} B A_1^{+\mathsf{T}}}$$

Insight

- Closed form solution using SVD
- Covariance matrix
  - exploits SVD of $A(\boldsymbol{l})$
  - Takes covariance matrix of residuals

$$\underline{\boldsymbol{g}} = A(\underline{\boldsymbol{l}})\boldsymbol{x}$$

    into account

✓ additional constraints enforce covariance matrix (e.g. determinant constraint of matrix F)

→ **Uncertainty measures for large class of methods**

Comparing the quality of methods

The task of comparing covariance matrices

Given:

Covariance matrix C

reference from method, from specification

Covariance matrix Σ

to be evaluated, from data or from method

Question:

Is method leading to Σ better than method with C

→ Various measures

Basic Idea

If both covariance matrices differ not too much then quotient

$$Q = \Sigma C^{-1}$$

should be close to unit matrix

Ratios of standard deviations

For diagonal covariance matrices

the ratios of the standard deviation for each $\widehat{\boldsymbol{x}}_u$

$$r_u := \sqrt{(\Sigma C^{-1})_{uu}} \approx \frac{\sigma^{\Sigma}_{x_u}}{\sigma^{C}_{x^2}}$$

Report (pretty standard)

- All ratios $\boldsymbol{r} = [r_1, ..., r_u, ..., r_U]^{\mathsf{T}}$
- The mean ratio $\overline{r} = \frac{1}{\sqrt{U}} |\boldsymbol{r}|_2 = \sqrt{1/U \sum_u r_u^2}$
- The maximum ratio $r_{\max} = \max_u (r_u)$

Ratios of standard deviations

Take correlations into account

→Replace squared ratios by eigenvalues

$$\lambda_u = \mu_u^2 = \lambda(\Sigma C^{-1})$$

= variances in direction of the eigenvectors

Report principle ratios

- All ratios
- The mean ratio
- The maximum ratio

$$\boldsymbol{\mu} = [\mu_1, ..., \mu_u, ..., \mu_U]^{\mathsf{T}}$$

$$\overline{\mu} = \frac{1}{\sqrt{U}} |\boldsymbol{\mu}|_2 = \sqrt{1/U \sum_u \mu_u^2}$$

$$\mu_{\max} = \max_u(\mu_u)$$

Interpretation

All measures

- Are unitless
- Can be interpreted as
  - Ratios of standard deviates
  - Loss/gain of accuracy
- Generally: using the eigenvalues
  - takes correlations into account
  - Is independent on coordinate system

Evaluating large vectors

Methods may provide large/very large vectors $\widehat{\boldsymbol{x}}$

(Bundle adjustment, SLAM, surfaces)

With reference values $\tilde{\boldsymbol{x}}$ we may use F-statistics

$$F = \frac{1}{U}(\widehat{\boldsymbol{x}} - \tilde{\boldsymbol{x}})^{\mathsf{T}} C^{-1} (\widehat{\boldsymbol{x}} - \tilde{\boldsymbol{x}}) \mid H_0 \sim F(U, \infty)$$

Usually, $F > 1$, therefore report accuracy loss

$$\boxed{\Delta F = \sqrt{F - 1} = \frac{\sigma_b}{\sigma_x}}$$

appr. relative bias induced by approximation

Examples: Accuracy loss During Point Cloud Registration

Example 1: Pose estimation of LIDAR sensor

Simplification of covariance matrix

Example: Pose estimation of LiDAR sensor

Given: reference coordinates of 3D points

Observed: I 3D points with LiDAR sensor

Sought: pose of sensor

Model: Similarity

Q: How much worse is

- estimation with global diagonal matrix, and
- estimation with local diagonal matrix

Compared to ML-solution?

Configuration

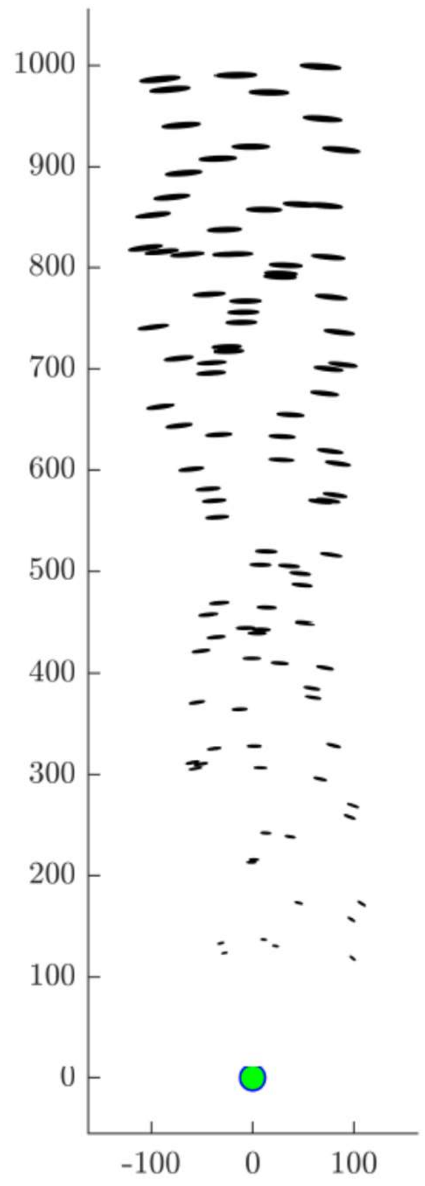
Visualization

- Position

- Points with uncertainty ellipsoids

→

- inhomogeneous,
- highly anisotropic



Configuration and models

Configuration

- $I = 100$ in box $200 \times 900 \times 20\ [\mathrm{m}]^3$ at distance $550\ [\mathrm{m}]$
- Leica sensor RTC360

$$\sigma_d = \sqrt{(0.001\ [\mathrm{m}])^2 + (10^{-5} d)^2} \quad \text{and} \quad \sigma_\alpha = 18''$$

Models

- Inhomogeneous anisotropic: $\Sigma^{(W)}_{X'_i X'_i} = f(d(X_i))$
- Inhomogeneous isotropic: $\Sigma^{(w)}_{X'_i X'_i} = \mathrm{tr}(\Sigma^{(W)}_{X'_i X'_i})/3\, I_3$
- Homogeneous, isotropic: $\Sigma^{(1)}_{XX} = \mathrm{tr}(\Sigma^{(W)}_{XX})/(3I)\, I_{3I}$

Loss in accuracy

Ratio of standard deviations

$$r_u^{mr} = \sigma_{\widehat{p_u}}^{(m)} / \sigma_{\widehat{p_u}}^{(r)} \quad \text{with} \quad m = 1, w, W$$

e.g. for $[s, R, \boldsymbol{t}]$

$$\boldsymbol{r}^{1W} = [2.68,\ 1.57,\ 1.68,\ 2.3,\ \mathbf{6.01},\ 3.88,\ 2.02]$$

Max ratios

$$r_{\max}^{1W} = 6.01\,, \quad r_{\max}^{wW} = 2.94\,, \quad r_{\max}^{1w} = 2.05$$

Mean ratios (including correlations!)

$$\overline{\mu}^{1W} = 2.59\,, \quad \overline{\mu}^{wW} = 1.80\,, \quad \overline{\mu}^{1w} = 1.82$$

→loss may be relevant

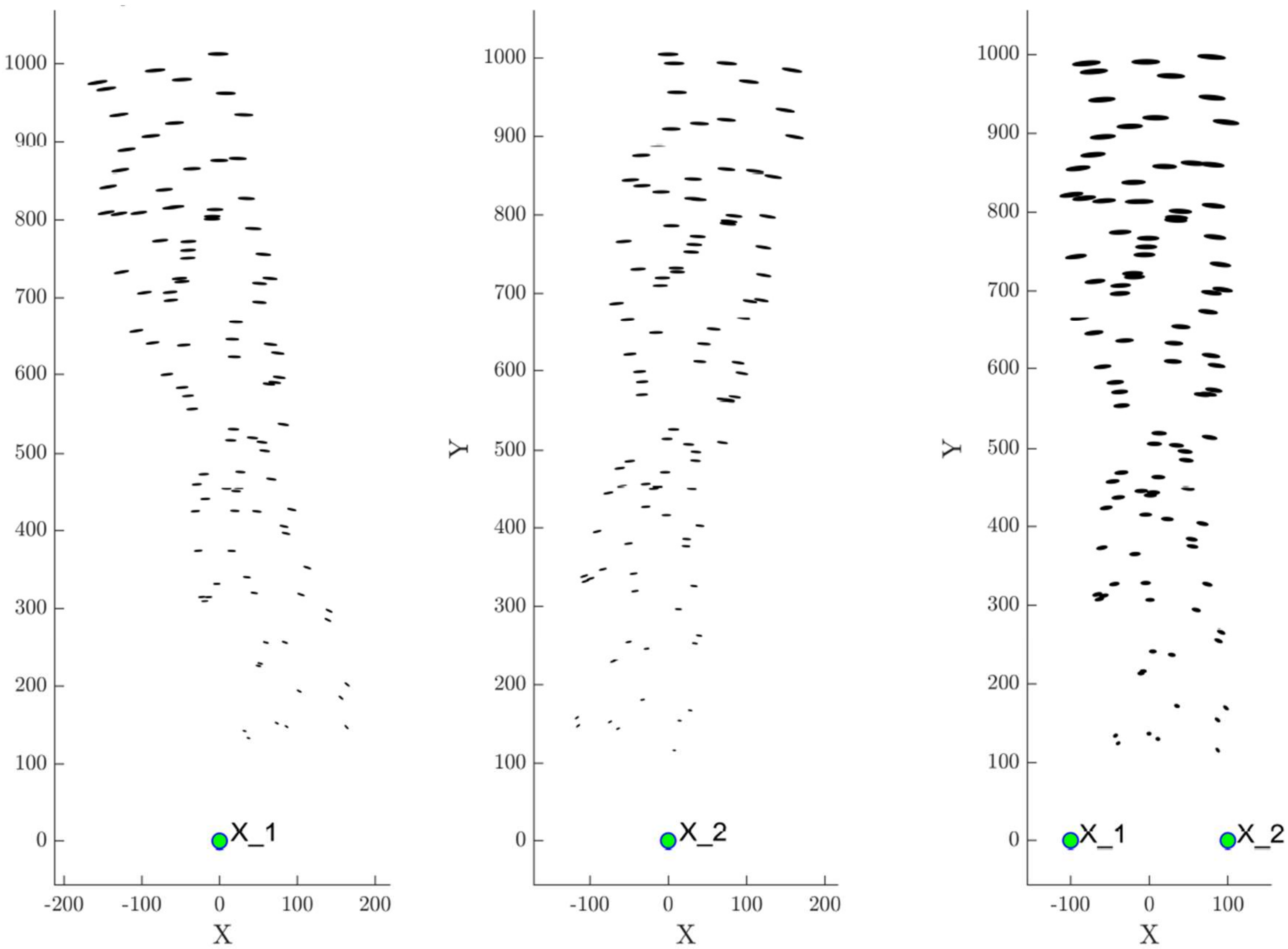
Example 2: Registration of LiDAR sensors

Simplification of covariance matrix

Example: registration of two LiDAR sensors

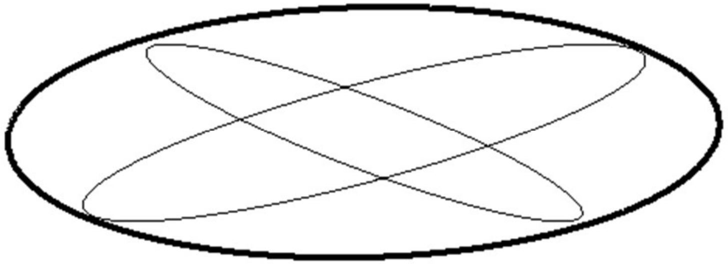
… same setup as before

Basis 200 m

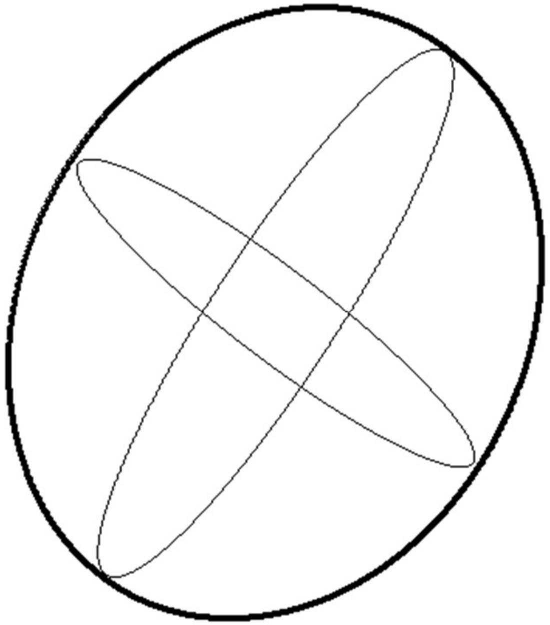


Example for adding CovM

$$\Sigma = \begin{bmatrix} 5.0 & 1.3 \\ 1.3 & 0.5 \end{bmatrix} + \begin{bmatrix} 3.0 & -1.1 \\ -1.1 & 0.5 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 2 & 2.7 \\ 2.7 & 4.4 \end{bmatrix} + \begin{bmatrix} 2.5 & -1.8 \\ -1.8 & 1.5 \end{bmatrix}$$



thicker



nearly isotropic

Basis 200 m: Loss in accuracy

Ratio of standard deviations of 7 parameters $\widehat{p}_u$

$$r_u^{mr} = \sigma_{\widehat{p}_u}^{(m)} / \sigma_{\widehat{p}_u}^{(r)} \quad \text{with} \quad m \in \{1, w, W\}$$

e.g.

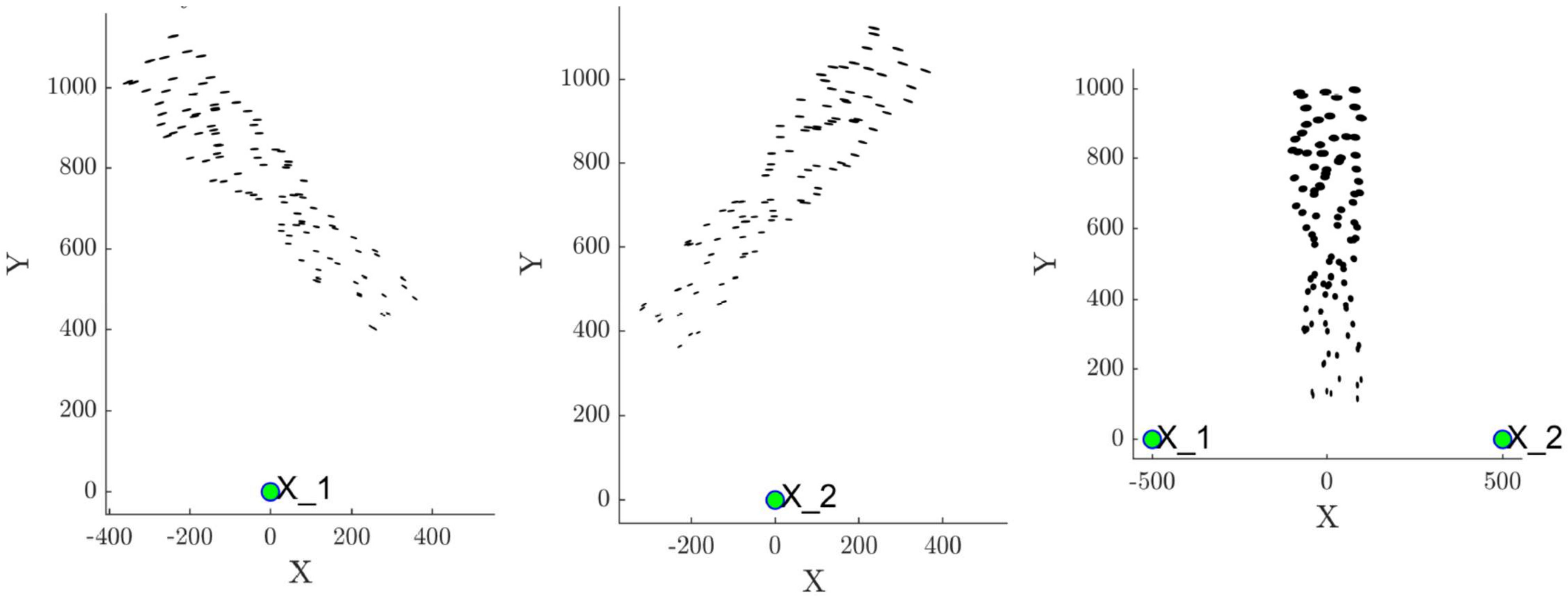
$$\boldsymbol{r}^{02} = [1.71,\ 1.67,\ 2.17,\ \mathbf{4.14},\ 2.18,\ 1.52,\ 2.29]$$

Max ratios

$$r_{\max}^{1W} = 4.14\,, \quad r_{\max}^{wW} = 2.32\,, \quad r_{\max}^{1w} = 1.87$$

→ loss may be relevant

Basis 1000 m



1000
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Y
X_1
-400
-200
0
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600
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Y
X_2
-200
0
200
400
X
1000
800
600
400
200
0
Y
X_1
X_2
-500
0
500
X

Basis 1000 m: Loss in accuracy

Ratio of standard deviations

$$r_u^{mr} = \sigma_{\widehat{p_u}}^{(m)} / \sigma_{\widehat{p_u}}^{(r)} \quad \text{with} \quad m \in \{0, 1, 2\} := \{1, w, W\}$$

e.g.

$$\boldsymbol{r}^{02} = [1.24,\ 1.20,\ 2.00,\ 1.72,\ 1.91,\ 1.20,\ 2.12]$$

Max ratios

$$r_{\max}^{1W} = 2.00\,, \quad r_{\max}^{wW} = 1.75\,, \quad r_{\max}^{1w} = 1.29$$

→ loss is smaller, still may be relevant

Example 3: Registration of LiDAR sensors

Comparing approximate methods

Task

Given: plane pairs $(\mathbf{A}, \mathbf{A}')_i, i = 1, ..., I$

Model:

$$\mathbf{A}'_i = \mathsf{M}^{-\mathsf{T}} \mathbf{A}_i \quad \text{with} \quad \mathbf{A}_i = \begin{bmatrix} \boldsymbol{n}_i \\ -s_i \end{bmatrix}, \; \mathbf{A}'_i = \begin{bmatrix} \boldsymbol{n}'_i \\ -s'_i \end{bmatrix}$$

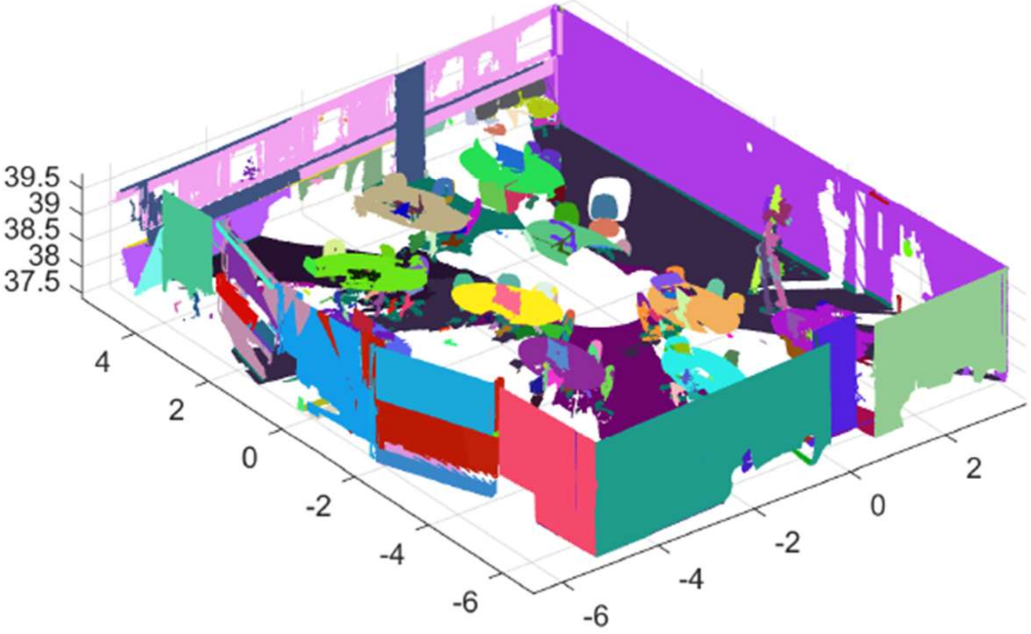
Constraints, linear in $(R, \boldsymbol{t})$

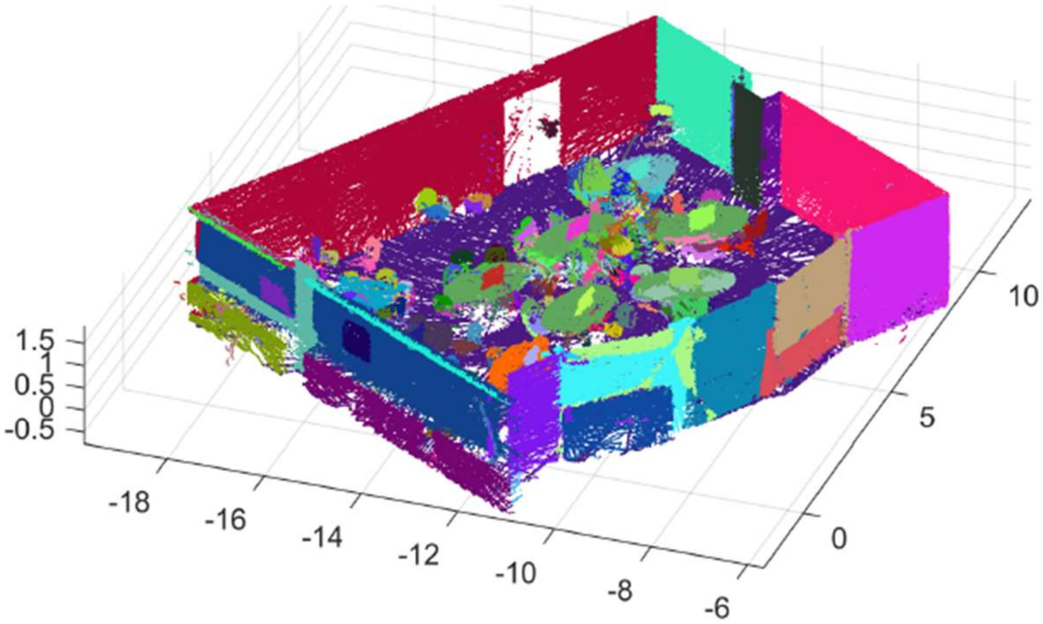
$$\mathbf{0} = R\boldsymbol{n} - \boldsymbol{n}' \quad \text{and} \quad \mathbf{0} = \boldsymbol{n}^{\mathsf{T}} \boldsymbol{t} - s + s'$$

→ Closed form solution by Khoshelham (2016)

Data sets

Faro and Zeb-Sensor, segmented into planes





Compare approximate methods with ML

Three approximate methods

ALG: Khoshelhams algebraic solution

$$\widehat{\boldsymbol{x}} = \min_{x,|x|=1} \boldsymbol{g}(\boldsymbol{x})^{\mathsf{T}} \boldsymbol{g}(\boldsymbol{x})$$

ALGw: ALG + weighted ALG (two-step)

$$\widehat{\boldsymbol{x}} = \min_{x,|x|=1} \boldsymbol{g}(\boldsymbol{x})^{\mathsf{T}} W_{gg} \boldsymbol{g}(\boldsymbol{x})$$

ML+1: ALG + 1 iteration ML (two-step)

Experimental setup

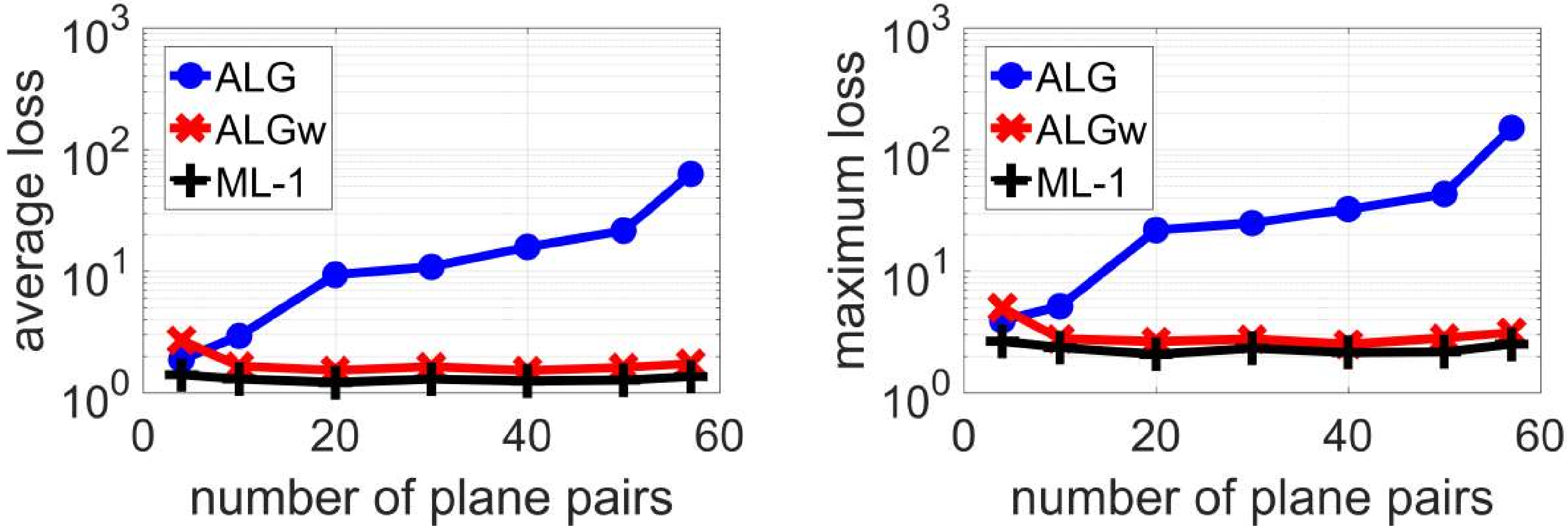
- Real data: Faro and Zeb-1 Sensor
- Segmentation into planes

  269 planes from 4.4 Mio., and 261 from 1.5 Mio. Points
- Accuracy: 1.2 mm and 25 mm
- Take estimated parameters/observations as true
- Contaminate according to accuracy
- Repeated sampling: $J = 100$
- 57 plane matches

Accuracy loss

Average loss $\overline{\mu}$ and maximum loss $\mu_{\max}$

Optimum value: no loss, bottom line ($1 = 10^0$)



→ Both upgraded approximate method perform well

Comparison with ICP

- ICP with point-to-point correspondences
- ICP with point-to-plane correspondences

Method	Average loss	Maximum loss
ML-1	1.24	2.04
ALGw	1.61	2.76
ALG	45.15	104.80
ICP-pl	3.66	6.09
ICP-pt	5.82	12.11

- With ALG quite high accuracy loss
- ICP-pl factor 2 better than ICP-pt
- Upgraded approximations: average loss < 60 %

Approximations in Bundle Adjustment

Approximations in Bundle Adjustment

- Using wrong Jacobians
- Neglecting correlations
- Omitting updates during iteration process

Bundle adjustment with constraints only

Constraints for each point

Relating image coordinates $\boldsymbol{x}$ to pose parameters $\boldsymbol{m}$

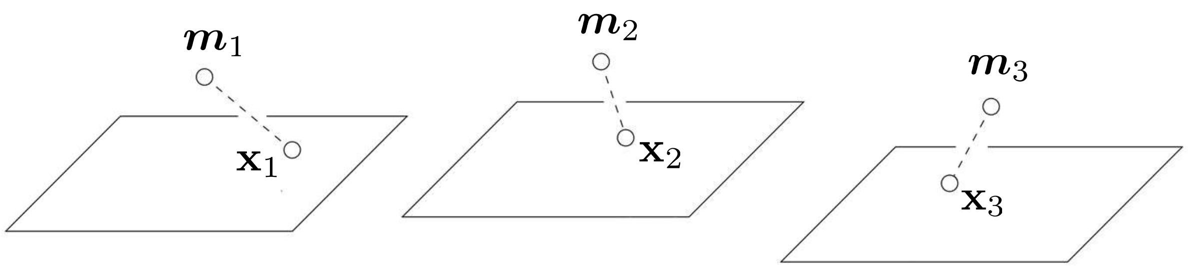
Two epipolar constraints

$$g_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{m}_1, \boldsymbol{m}_2) = 0\,, \quad g_2(\boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{m}_2, \boldsymbol{m}_3) = 0$$

One trifocal constraint

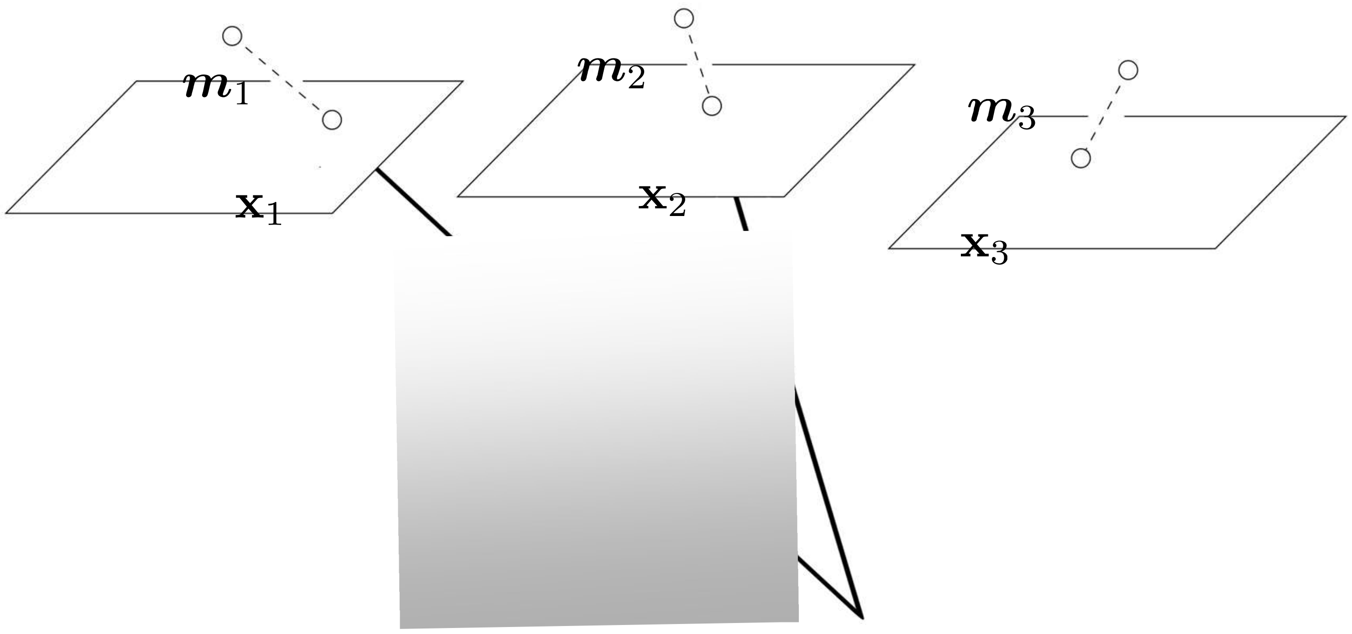
$$g_3(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3) = 0$$

Epipolar & Trifocal Constraints



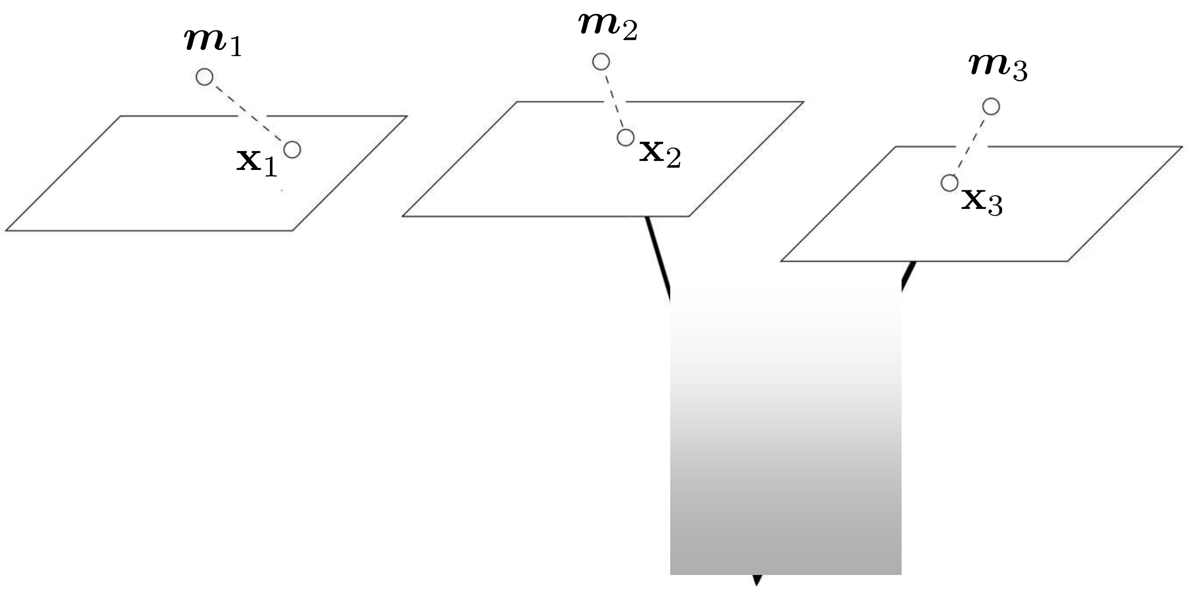
Epipolar & Trifocal Constraints

1.Epipolar constraint $g_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{m}_1, \boldsymbol{m}_2) = 0$



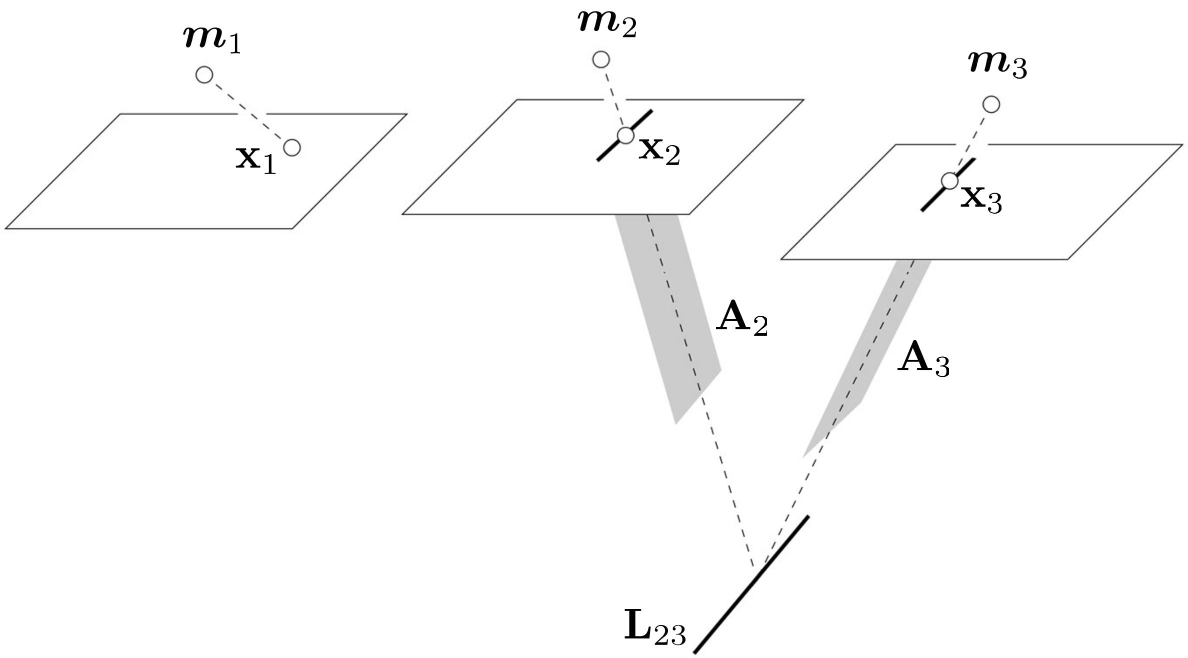
Epipolar & Trifocal Constraints

1. Epipolar constraint $g_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{m}_1, \boldsymbol{m}_2) = 0$
2. Epipolar constraint



Epipolar & Trifocal Constraints

1. Epipolar constraint $g_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{m}_1, \boldsymbol{m}_2) = 0$
2. Epipolar constraint $g_2(\boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{m}_2, \boldsymbol{m}_3) = 0$
3. Trifocal constraint $g_3(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3) = 0$



Normal equation system

Normal equations $N \Delta \boldsymbol{x} = \boldsymbol{n}$

With

$$N = A^{\mathsf{T}} \underbrace{\Sigma_{nn}^{-1} A}_{=C} \qquad \boldsymbol{n} = A^{\mathsf{T}} \underbrace{\Sigma_{nn}^{-1} \Delta \boldsymbol{l}}_{=\boldsymbol{c}} \qquad \Sigma_{nn} = B^{\mathsf{T}} \Sigma_{ll} B$$

Both matrices B and N sparse, not diagonal

→

- Solve two equation systems

$$B^{\mathsf{T}} \Sigma_{ll} B \ [C, \boldsymbol{c}] = [A, \Delta \boldsymbol{l}] \quad \text{and} \quad A^{\mathsf{T}} C \ \Delta \boldsymbol{x} = A^{\mathsf{T}} \boldsymbol{c}$$

- Or, make approximations

Analysis of approximattions

A: evaluate Jacobians at $\boldsymbol{l}$, instead at $\widehat{\boldsymbol{l}} = \widehat{\boldsymbol{y}}$

(save estimating $\widehat{\boldsymbol{l}} = \widehat{\boldsymbol{y}}$)

B: Neglect correlation in W_{nn}

$$W_{nn}^{[B]} = \mathrm{Diag}(W_{nn})$$

C: A and B

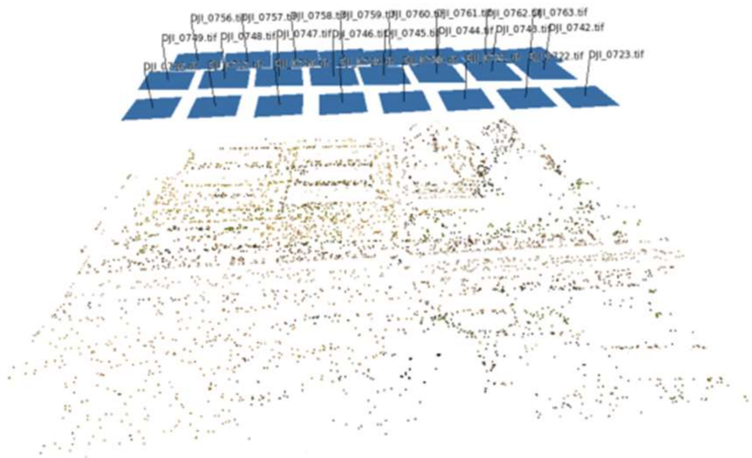
D: use $W_{nn}^{[B]}$ and fix it after 1st iteration

(result also depends on approximate values)

Data sets

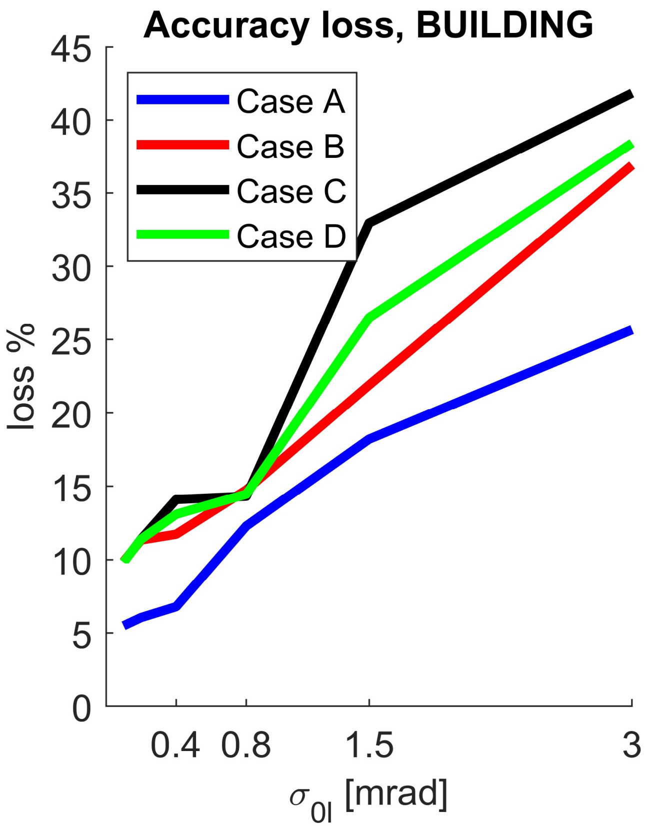
name	image size	focel length	distance
BUILDING	5 MPixel	1589 pixel	approx 15 m - 60 m
FIELD	12 MPixel	2347 pixel	100 m

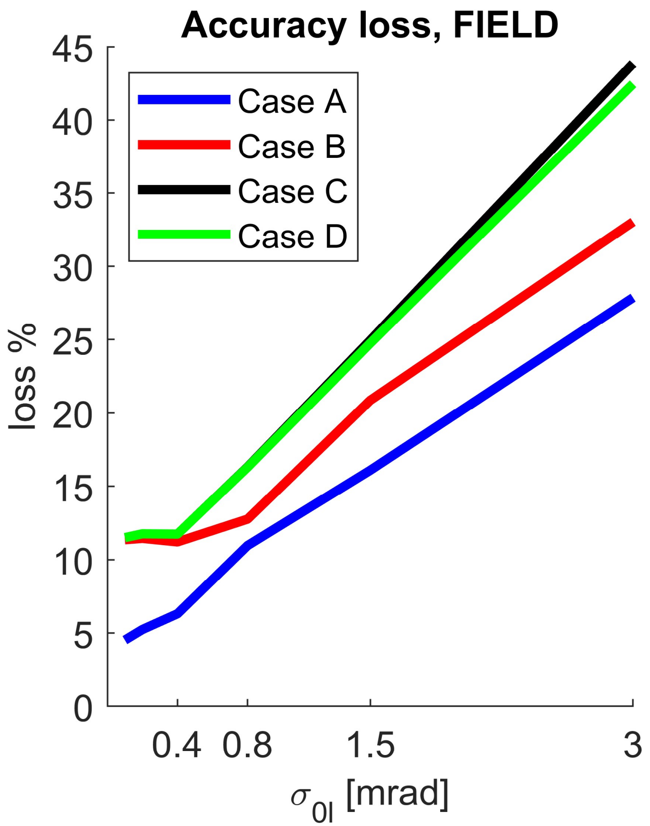
Table 5: Datasets with some characteristics



Accuracy loss

Accuracy loss ΔF in % as a function of noise σ_{0l}





Results: effect of observational noise

Input: Noise level moderate (0.2 to 6 pixels)

Accuracy losses:

- Effect of approximate Jacobian (A): 5 % to 25 %
- Effect of neglected correlations (B): 10 % to 35 %
- Combined effect: 10 % to 45 %
- Fixing approximate weight matrix: 10 % to 45 %

Results: effect of approximate values (D)

Scatter of approximate values:

Relative pose errors

Accuracy loss:

σ_{0x} [mrad]	1.0	3.0	10.0	30.0	100.0
σ_{0x} [°]	0.06	0.17	0.57	1.72	5.73
Building					
Case D	9.82	10.52	17.16	20.97	31.44
Field					
Case D	11.50	13.28	14.52	16.94	25.36

→ Stays below 30 %

Relevance of analysis and results

Relevance of analysis and results

The user’s perspective (consumer)

The author’s perspective (producer)

The user's perspective

- Approximate methods may be acceptable
- Choice
  - Approximate method with precise instrument
  - Rigorous method with less precise instrument
- Need for constant/predictable CPU-time?

  robotics, interactive systems
- Does the user need uncertainty information?

  Some approximations do not provide this

  Cramer-Rao bound as criterium for optimal method

The author's perspective

Authors motivation to publish methods

- Establish a reference method for a unsolved problem
- Provide a more efficient solution
- Provide analysis of a method's accuracy
- Provide approximate method with specified accuracy
- Consider potential user's requirements

Requirements for publication

- Improvement: stdv -20 %, CPU-time -30 % (Moore)
- Provide specification sheet for performance

Closure

Closure

- Notion of rigorous method needs to be specified
- ML estimation is golden standard
- Upgrade of approximate method
  - Information on uncertainty of approximate parameters
  - Add a single ML step → nearly optimal results
- Examples show large variety of accuracy losses
  - Between 0 and infinity
  - Mostly moderate, below factor 2
- The user, not the author, decide on usefulness

Outlook

Are the ideas generalizable to deep learning methods?

Ideas are based on

- Simple distributions
- Linearization
- Algebraic short cuts (general and linear algebra)

= closed world assumption

→

Transfer of ideas to deep learning remains a challenge

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